Lesson 3

# Thermomechanical Measurements for Energy Systems (MENR)

## Measurements for Mechanical Systems and Production (MMER)

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# • ACCURACY (Precision)

## Attitude of a measuring instrument to give the «true value» $a_v$ of a physical quantity !

Engineers and technicians always wished the instruments being able to perform ***precise measurements*** (with *high accuracy*).
However it is *unavoidable to make errors* during measurement processes, so we must define a convenient way to express *how good or how bad* our measurements ***a*** are !
A first simple way could be expressing the "distance" between the *measurement a* and the *true value* $a_v$
$\varepsilon = |a - a_v|$ which is the ***absolute error*** of the measurement.
However, there is a more convenient way in engineering to express the *accuracy of a measurement*:

$$\varepsilon(\%) = \frac{a - a_v}{a} \times 100$$

which is the ***relative error*** of the measurement (expressed in %), and gives an immediate and intuitive idea of the measurement accuracy !
If we knew the true value $a_v$ , we were basically all set and we could go ahead to analyze the next characteristics.

Unfortunately, the **true value** of a physical quantity is generally **unknown** and **unknowable** !

To get ***EXACT INFORMATION*** about the *measurand* we should have available an ***exemplar measurement method*** !

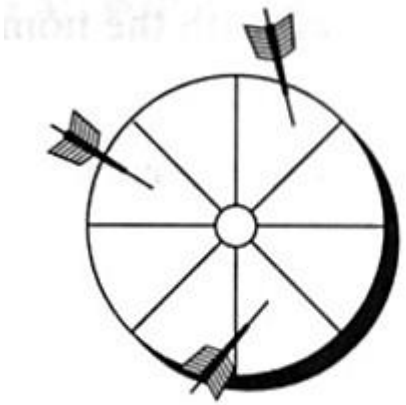
A measurement is therefore a *technical procedure* that strives to get *as close as possible to the* ***truth*** of the natural world and/or of the technology ! … But, it can never succeed …

We can only *minimize* the unavoidable ***uncertainties*** $\varepsilon_a$ **:** $A = (a \pm \varepsilon_a) \cdot U$ trying to get $\varepsilon_a \rightarrow 0$

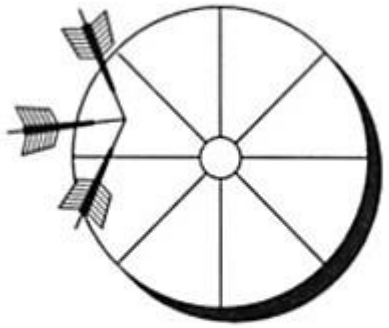
There are *two* main types of error that can be done during a measurement:

1. **Systematic (BIAS) errors**
2. **Random (PRECISION) errors**

***Measurement accuracy*** results from a combination of these two errors !



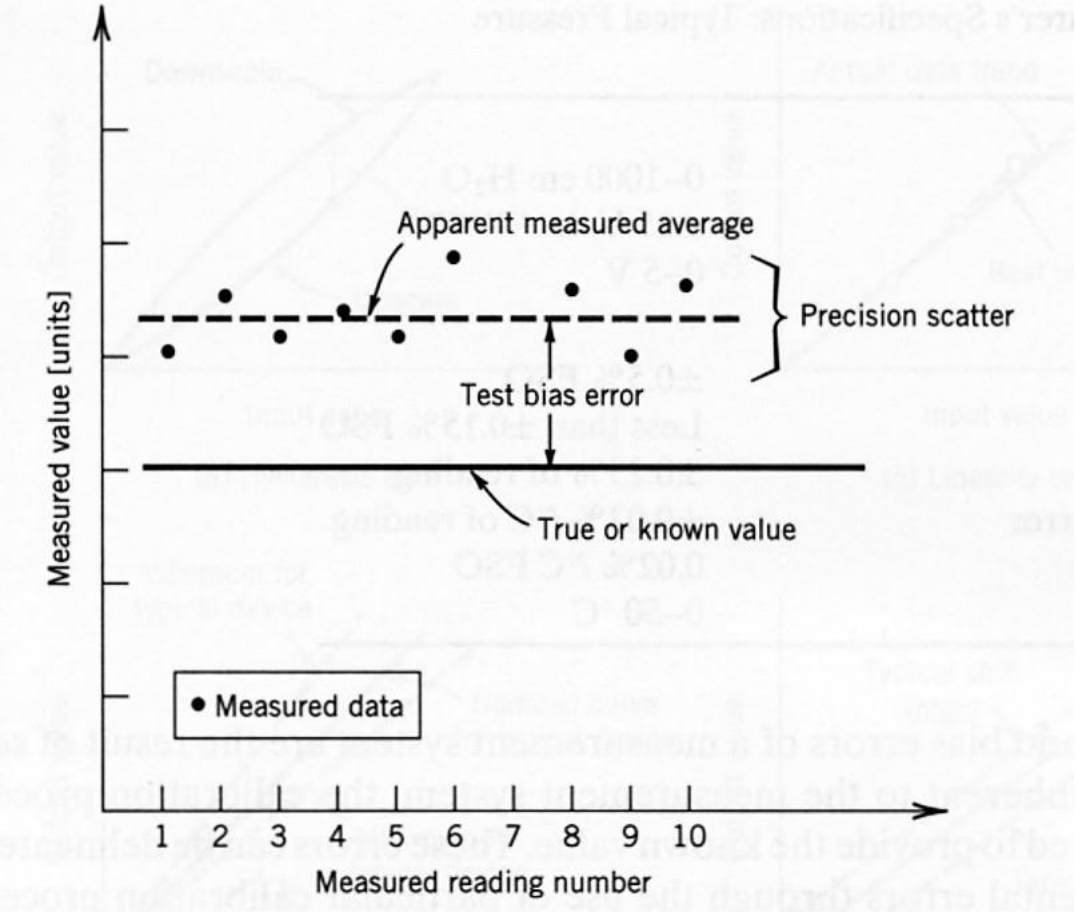
(*b*) Bias and precision errors lead to poor accuracy



(*a*) High repeatability gives low precision error but no direct indication of accuracy

(*b*) High accuracy means low precision and bias errors

***Bias errors*** are often "macroscopic" errors because they have a «technical origin» (instrument malfunction, bad conditions of use, strong external noise that the instrument can not reject, instrument out of calibration). Bias errors CAN be discovered by a calibration procedure and, once highlighted, they MUST be eliminated !




***Random errors*** are generally small "scatter errors", that can not be avoided during a measurement and for which is impossible to find an exact reason !

*Random errors* can be studied after the *bias error* has been corrected for, or (better) eliminated.
Therefore the *sum of all random errors* is, sometimes, also called ***residual error***.

The *residual error* always depends on *unknown small random reasons*, therefore it is much better referred with the word ***uncertainty*** !

According to the way we evaluating it, there are only two different types of ***UNCERTAINTY***, which are classified by the international «***Guide to the expression of uncertainty in measurement***» ed. JCGM 100: 2008 (***GUM***)
http://www.bipm.org/en/publications/guides/gum.html

1. Type A uncertainties: evaluated by a ***statistical analysis*** of series of observations (measurements)
2. Type B uncertainties: evaluated by a ***pool of comparatively reliable information***

The *uncertainty* of the result of a measurement reflects the *"lack of exact knowledge"* of the measurand value.
The result of a measurement, after correction for recognized *systematic effects*, is still only an *estimate of the value of the measurand* because of the uncertainty arising from random effects and from imperfect correction of the result for systematic effects.
In practice, there are many possible *sources of uncertainty* in a measurement, including:

a) incomplete definition of the measurand;
b) imperfect realization of the definition of the measurand;
c) non-representative sampling — the sample measured may not represent the defined measurand;
d) inadequate knowledge of the effects of environmental conditions on the measurement or imperfect measurement of environmental conditions;
e) personal bias in reading analogue instruments;
f) finite instrument resolution or discrimination threshold;
g) inexact values of measurement standards and reference materials;
h) inexact values of constants and other parameters obtained from external sources and used in the data-reduction algorithm;
i) approximations and assumptions incorporated in the measurement method and procedure;
j) variations in repeated observations of the measurand under apparently identical conditions.

These sources are not necessarily independent, and some of sources a) to i) may contribute to source j).
Of course, an *unrecognized systematic effect* cannot be taken into account in the evaluation of the uncertainty of the result of a measurement but contributes to its error !

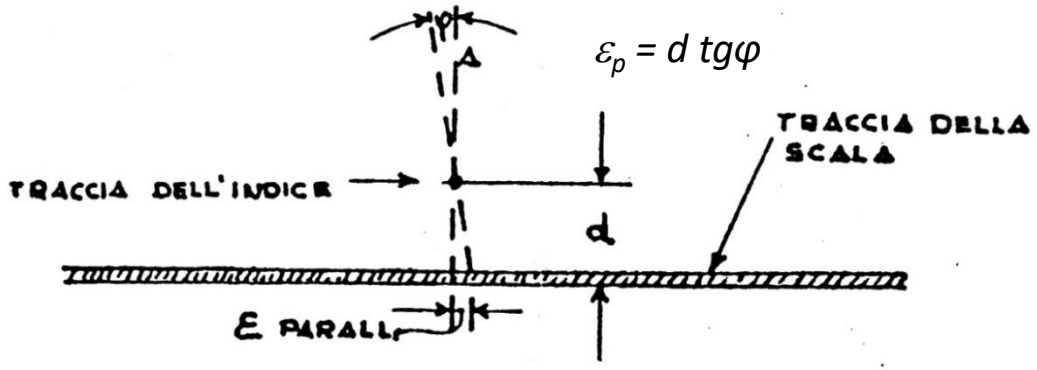
For simplicity, we start the study of *uncertainty estimation* with the **type B uncertainty**:

For an estimate $x_i$ of an input quantity $X_i$ that has not been obtained from repeated observations, the associated *estimated variance* $u^2(x_i)$ or the ***standard uncertainty*** $u(x_i)$ is evaluated by *scientific judgement* based on all of the *available information on the possible variability of* $X_i$ .

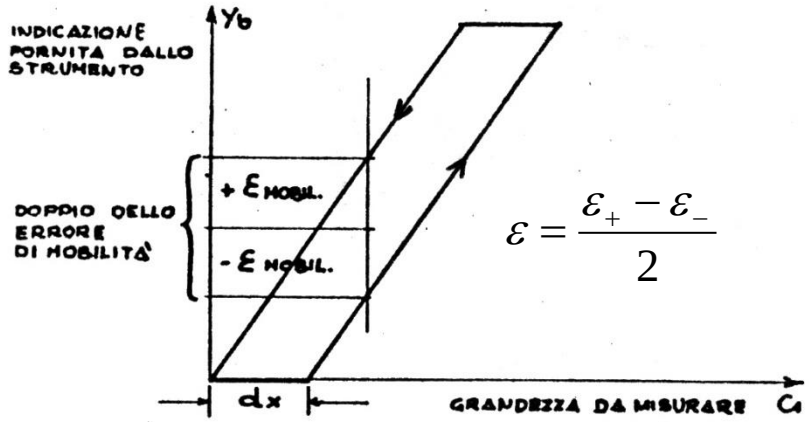
The pool of information may include :

- previous measurement data;
- experience with or general knowledge of the behaviour and properties of relevant materials and instruments;
- manufacturer's specifications;
- data provided in calibration and other certificates;
- uncertainties assigned to reference data taken from handbooks.

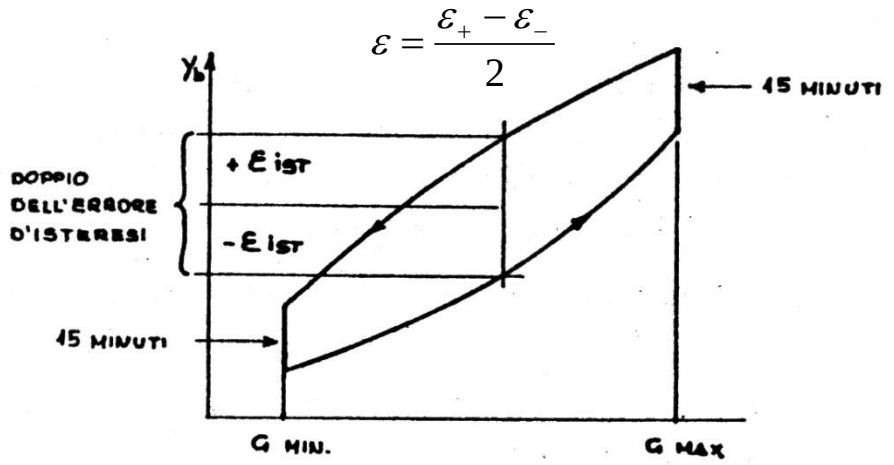
Examples of typical errors in mechanical measurements that can contribute to type B uncertainties :




***Reading error*** for analogic instruments




***Mobility error*** for mechanical instruments

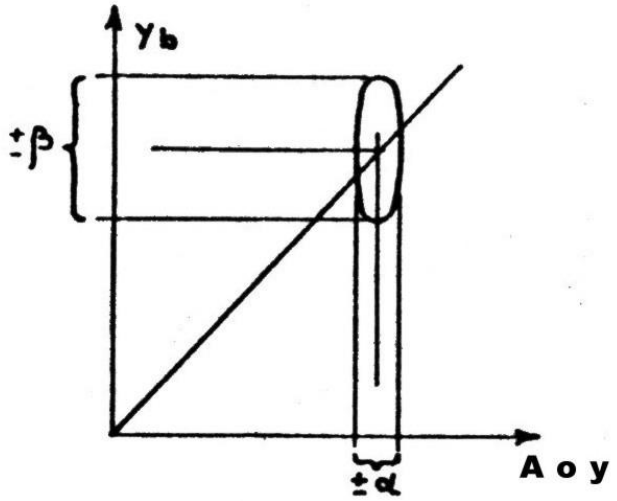



***Hysteresis error*** for mechanical instruments

***Fidelity error***: due to the *disturbing* and *noise factors* as temperature, humidity, electromagnetic fields, mechanical vibrations, atmospheric pressure, non-inertial reference frame ...
It quantifies the sensitivity of the instrument to *external interferences*, and it has to be assessed using multiple repeated measurements over time, by changing the position and the conditions of use of the instrument while keeping the input variable strictly constant …

***Zero shift error***: due to *loss of calibration* of the mechanical components (recording springs) or electrical components (trimmer), or to the aging of the electronic components (resistors and capacitors) …

***Calibration and standard references errors***: due to the errors that are done *while carrying out a calibration* procedure or due to the *inherent uncertainties of the reference standards* used for the calibration !




±α are the *standard references errors*
±β are the errors done *while tracing* the calibration curve

The total error is a quadrature sum: $\varepsilon_t = \sqrt{\alpha^2 + \beta^2}$ and can be represented with the ellipsoid show in the figure.

An older way of expressing the *type B uncertainties* [in %] was with the ***precision class*** parameter: $\frac{\sqrt{\sum_i \varepsilon_i^2}}{full\_scale}$
where $\varepsilon_i$ is each *error* or *uncertainty source* one is able to isolate and estimate

Example:

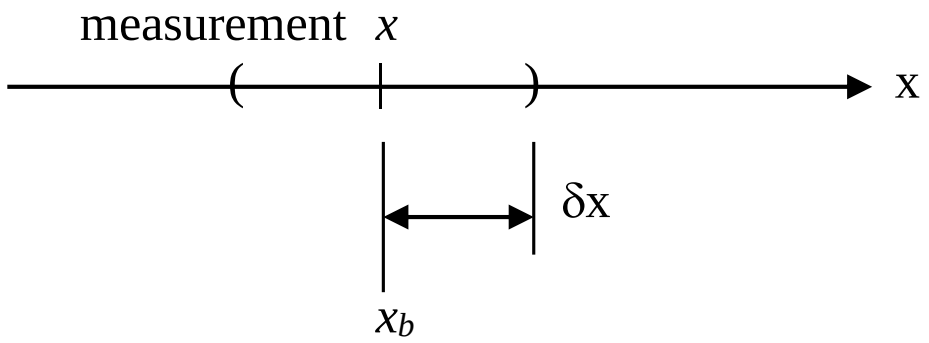
a *class 0.5* dynamometer with a full scale of 100 N makes an absolute error of 0.5N (0.5% relative error) when measuring 100N the *class 0.5* dynamometer makes the same absolute error of 0.5N when measuring 5 N, which now is a relative error of 0.5N / 5N = 0.1 = 10 %.
An advice: never use an instrument for which a precision class is declared at the beginning of the span !

We now proceed to explain how to *estimate* the **type A uncertainty** :

Type A uncertainty estimation has a completely different approach → it does NOT even care about *identifying error sources !*
The method is based on a *statistical data post processing* and, therefore, we need a sufficient number *n* of repeated measurements with the *measurand X held* ***strictly constant*** !


$x_i$ is the $i_{th}$ measurement of the *measurand X*

The final goal is NOT even to find the *true value* $x_v$ but a certain ***range of values*** within which the true value could reasonably be found ! This range could be written with $x = x_b \pm \delta x$ where $x_b$ is the best representation of the true value (x best) and $\delta x$ is a width parameter that might be *representative of the uncertainty* !




Because our *n* measurements $x_1$, $x_2$ … $x_n$ will be *very similar* each other but *not equal* between them, which of them could possibly be the *best representation* of our *n* measurements ?

$$x_b = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$ The ***mean value*** of our data: the *n* measurements !

So far, there is NO theoretical justification for our choice; its only a *very reasonable* choice, so reasonable that I will try to apply it also to find a *quantitative expression* for the width parameter *δx* :

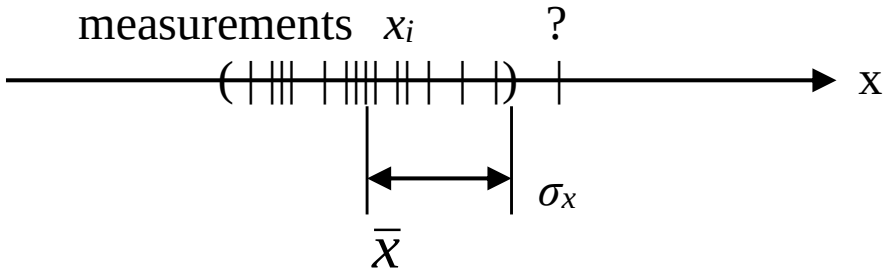
$$d_i = x_i - \bar{x}$$ It's the "*deviation*" of the generic measurement $x_i$ from the mean $\bar{x}$
Can we calculate the *mean value* for this deviation ?

$$\bar{d} = \frac{1}{n}\sum_{i=1}^{n} d_i = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i - \frac{1}{n}\sum_{i=1}^{n}\bar{x} = \bar{x} - \frac{1}{n} n \cdot \bar{x} = 0$$

NO ! $\bar{d} = 0$ because the deviations are "*equidistant*" from the mean value !

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2} = \sigma_x$$ Therefore, the ***Standard Deviation*** (of the *population*) has been proposed !

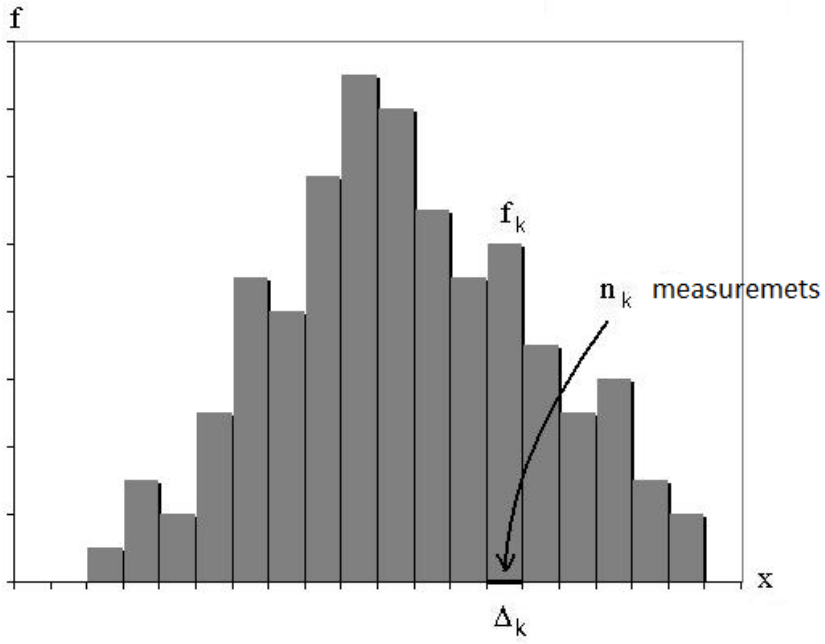
$$\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2} = \sigma_x$$ or, with only a few measurement points, the ***Standard Deviation*** (of the *sample*) !
which produces $\sigma_x = 0/0$ instead of $\sigma_x = 0$ when n → 1




Now we can express our measurement $x_i$ with the *best representative* $\bar{x}$ and with the *width parameter* $\sigma_x$ !

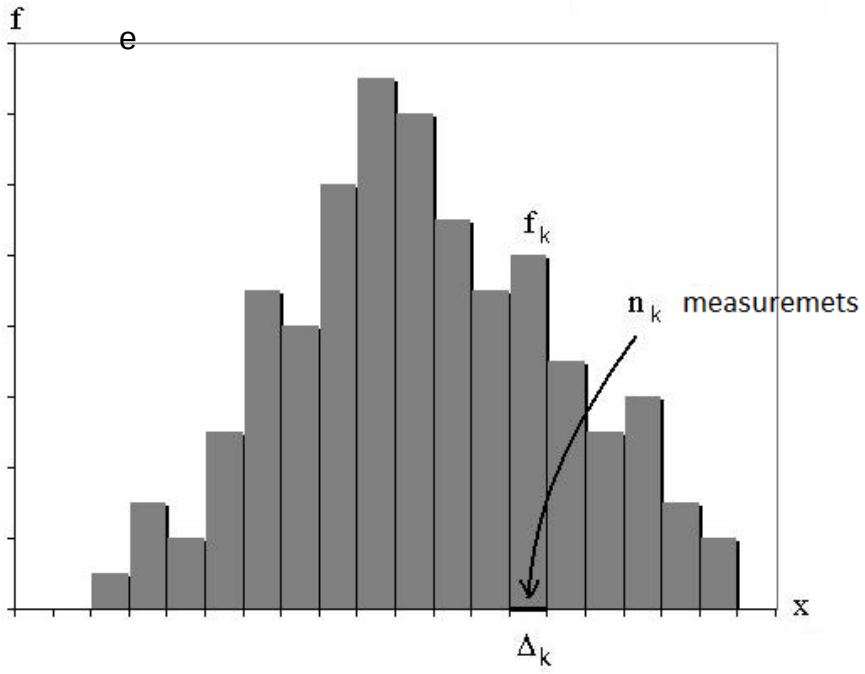
$$x_i = x_{n+1} = \bar{x} \pm \sigma_x$$

We have now to ask ourselves:
*How much* do we *trust* these choices ? Is *every measurement* falling *"inside"* the range we just determined *?*
Or, better, what ***CONFIDENCE*** do we place in this range of values ?




- To answer these question we have first to construct the figure on the left, where we divided the "range" where the measurements fell in *k* small intervals $\Delta_k$ with at least one measurement inside each interval $\Delta_k$ .
- The more measurements $x_i$ we have available the more (and smaller) intervals we can choose.
- The height $f_k$ of each rectangle is proportional to the number of measurements inside each interval $\Delta_k$ .
- This figure is called the ***Frequency Histogram.***

## *FREQUENCY HISTOGRAM*




$n_k$ → *number of measurements* (***observations***) that fall into the interval $\Delta_k$

$f_k = \frac{n_k}{n}$ → ***frequency of the observations*** (*measurements*) that fall into the $\Delta_k$ interval

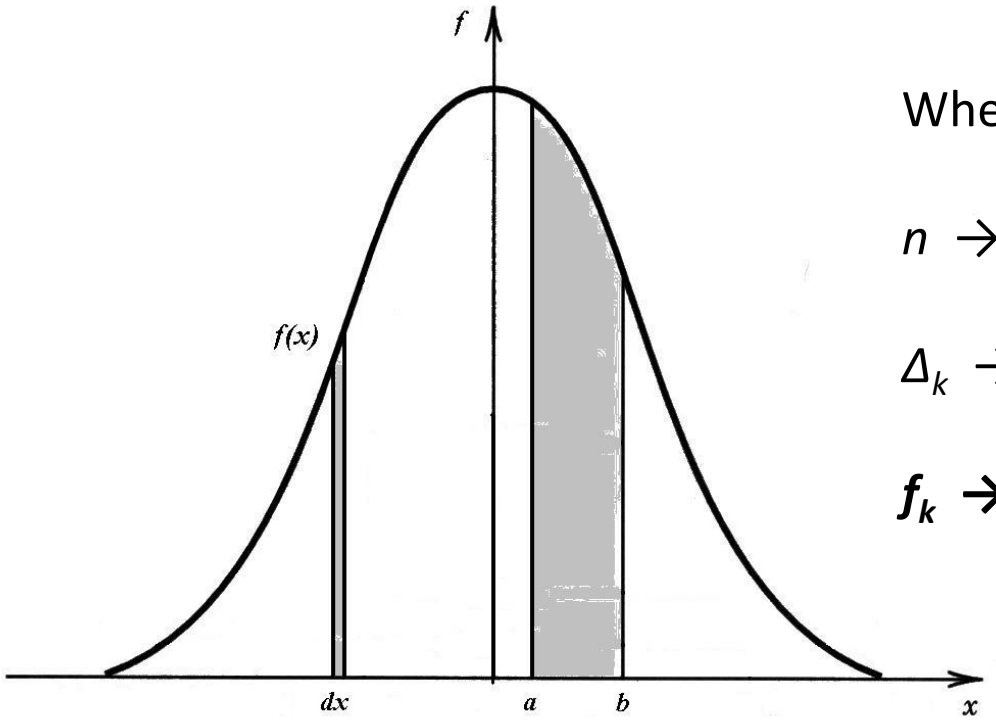
Obviously: $\sum_k n_k = n$ and $\sum_k f_k = 1$

The ***frequency histogram*** is "*normalized*" by definition !

The area $f_k \cdot \Delta_k$ represent the *measurement fraction* that falls within the $\Delta_k$ interval.

Note that it is possible to express the mean value of all the *n* measurements as: $\bar{x} = \frac{\sum_i x_i}{n} = \frac{\sum_k x_k n_k}{n} = \sum_k x_k f_k$
where $x_k$ is the *best representation* of $x_i$ inside the $\Delta_k$ interval

We now think ideally to ***bring the number of measurements n → ∞*** … in this situation we can think of an "*enormous*" number of intervals $\Delta_k$ which are also going to be "*smaller and smaller*" …
*The stepwise* ***curve of the frequency histogram*** *becomes a smooth* ***limit distribution curve f(x)***




Where:

$n \rightarrow \infty$

$\Delta_k \rightarrow dx$

$\boldsymbol{f_k \rightarrow f(x)}$

$f(x)dx$ is the *measurement fraction* that falls within the *infinitesimal dx interval*

$\int_a^b f(x)dx$ is the *measurement fraction* that falls within the *finite interval (b-a)*

Because it is $\int_{-\infty}^{\infty} f(x)dx = 1$

the limit distribution curve is also *normalized* !

Only for $n \rightarrow \infty$ $\quad f(x) \equiv p(x)$

**Frequency**
(experimental results)

**Probability**
(theoretical model )

But, now, which curve describes at best the limit distribution curve *f(x)* ?

It's the well-known ***Normal distribution curve*** or ***Gaussian curve*** or ***bell curve*** …

$$f(x) = e^{-\frac{(x-X)^2}{2\sigma^2}}$$

$$f(x) = e^{-\frac{(x-X)^2}{2\sigma^2}}$$

where ***X*** is the *"mean"* or the *true value* of *x* (in fact we have now $n \to \infty$ measurements) and ***σ*** is a *"width parameter"* of the function

Applying the normalizing condition: $\int_{-\infty}^{\infty} f(x)dx = 1$ we get

$$f_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-X)^2}{2\sigma^2}}$$

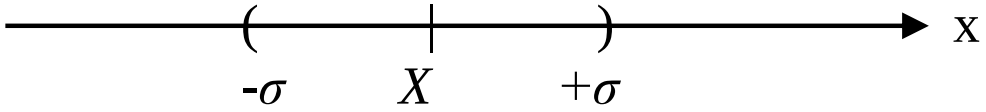
If we calculate the *mean value* for the function $f_{X,\sigma}(x)$ : $\bar{x} = \int_{-\infty}^{\infty} x \cdot f_{X,\sigma}(x)dx = X$ we got the ***mean*** !

If we calculate the *mean squared difference* for $f_{X,\sigma}(x)$ : $\sigma_x^2 = \int_{-\infty}^{\infty} (x-\bar{x})^2 f_{X,\sigma}(x)dx = \sigma^2$ we got the ***variance*** !

which is the ***squared standard deviation …***

Therefore, "theoretically" the mean ***X*** is the ***true value*** of the measurement and the width ***σ*** is the ***standard deviation,*** calculated for the ideal case of $n \to \infty$ measurements !!!

Finally, if we refer to the mathematical model $f_{X,\sigma}(x)$, we can express the *measurement x* with :




$$x = X \pm \sigma$$

We still have to understand what ***confidence*** we give to the *width parameter* ***σ*** …

The good news is that with a "mathematical model" we can calculate the integrals between two specified limits:

$$p(x \in \pm\sigma) = \int_{-\sigma}^{+\sigma} f_{X,\sigma}(x)dx$$

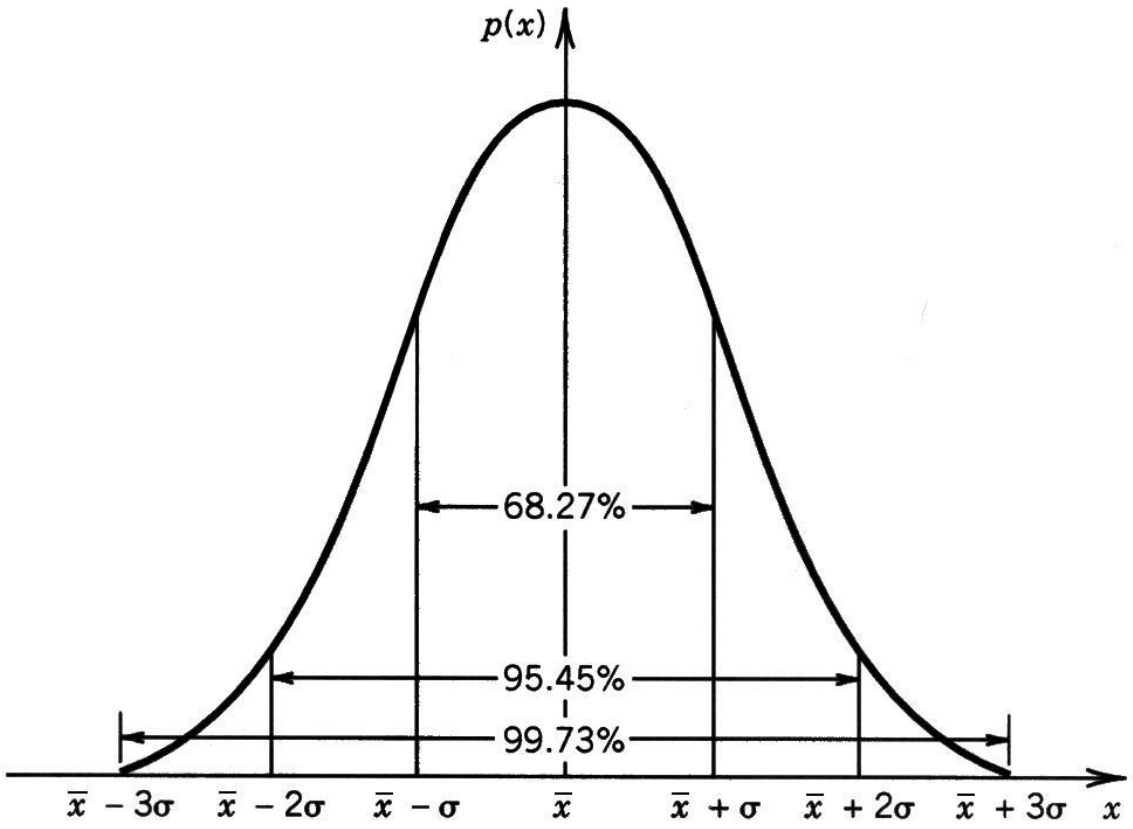
$$p(x \in \pm 2\sigma) = \int_{-2\sigma}^{+2\sigma} f_{X,\sigma}(x)dx$$

$$p(x \in \pm 3\sigma) = \int_{-3\sigma}^{+3\sigma} f_{X,\sigma}(x)dx$$

And they will represent the ***probability*** one measurement **x** has (one of the ∞ we have at disposal) to fall within the range ± σ, or ± 2σ, or ± 3σ around the *true value* ***X***

$$p(x \in \pm k\sigma) = \begin{cases} 68{,}27\% \rightarrow k = 1 \\ 95{,}45\% \rightarrow k = 2 \\ 99{,}73\% \rightarrow k = 3 \end{cases}$$

The *factor* ***k*** is named in the GUM as ***coverage factor*** and leads to the definition of ***expanded uncertainty*** …




Therefore, writing $x = X \pm \sigma$ means there is a 68.3 % "probability" to find a *measurement x* within the uncertainty range $\bar{x} - \sigma$ and $\bar{x} + \sigma$ .

In other words, we gained a ***confidence*** of 68.3% of finding our measurements within the range $\bar{x} \pm \sigma$ .

The geometrical representation of the ***probability*** $p(x \in \pm\sigma)$ discussed above, is shown left on the gaussian distribution curve, also for *coverage factors* of 2 and 3, and is the *area* underlying the curve.

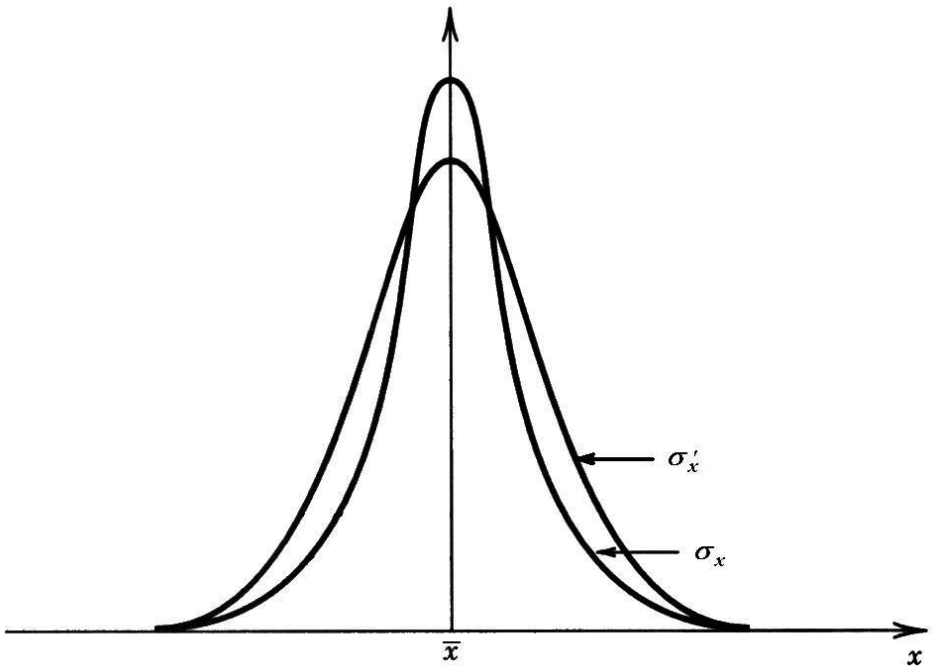
It is very important to understand the two following concepts:

$x = X \pm \sigma$ is a ***probability*** and can be *calculated before* doing a single measurement (having the function) !

$x = \bar{x} \pm \sigma_x$ is a ***statistics*** and can calculated *only if we have data* available (after doing the measurements) !

Keep always in mind that the *confidence intervals* and their *probability values* have been determined from a theoretical model or considering *∞ measurements available*, which ***NEVER*** happens in the real world, where we ***ALWAYS*** have only a *limited number n of measurements available* !

Before applying the results of the *probability theory*, we have to be reasonably sure that the distribution of our measurement is actually a Gaussian one (***$\chi^2$ test***) otherwise, we should rather try applying other statistical distributions (***t Student***) …




Because, by increasing the number *n* of the measurements the standard deviation $\sigma_x$ does not change much, $x = \bar{x} \pm \sigma_x$ is a good way of expressing the *uncertainty of the instrument* or *of the measurement method*.

Observing the *shape* of the measurement distributions, it is therefore immediate *establishing* or *comparing* the accuracy of different measuring instruments !!

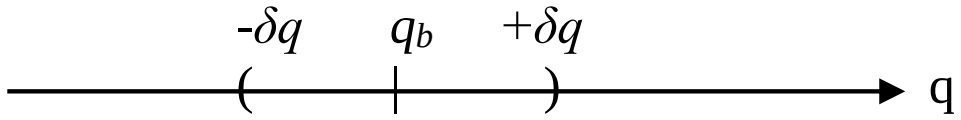
Being $\bar{x} \neq X$ we still have to understand if it is possible to get from our *n* measurements $x_1\ x_2 \ldots x_n$ any information about the ***accuracy of the measurement*** ???

To answer to this *ambitious question* we have to make a step back and …

get a few basic concepts of the ***Error Propagation Theory*** …

Consider a physical quantity *"q = x + y"* that can be expressed by the ***sum*** of two other primary physical quantities:

$x = x_b \pm \delta x$ and $y = y_b \pm \delta y$ We wish to find: $\boxed{q = q_b \pm \delta q}$

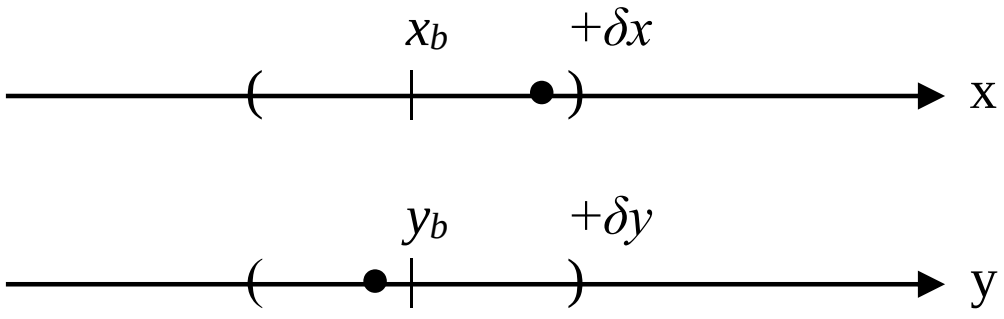



with $q_b = x_b + y_b$

$x_b + y_b + (\delta x + \delta y)$ which might represent the *upper limit* of the width *δq*

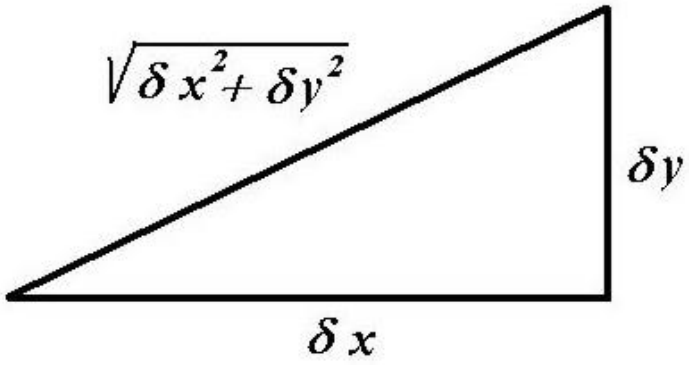
$x_b + y_b - (\delta x + \delta y)$ which might represent the *lower limit* of the width *δq*

We might, therefore, think that $\delta q = \delta x + \delta y$ however, if the errors that lead to *δx* and *δy* are *independent*, the choice $\delta q = \delta x + \delta y$ is an ***overestimation*** of the *uncertainty* for q …




In fact, to actually have $\delta q = \delta x + \delta y$ we should *always underestimate* or *overestimate at the same time* the measurement of *x* and *y* , which would imply an "underling low" or a "correlation" between the two variables, making them somehow *dependent* on each other !

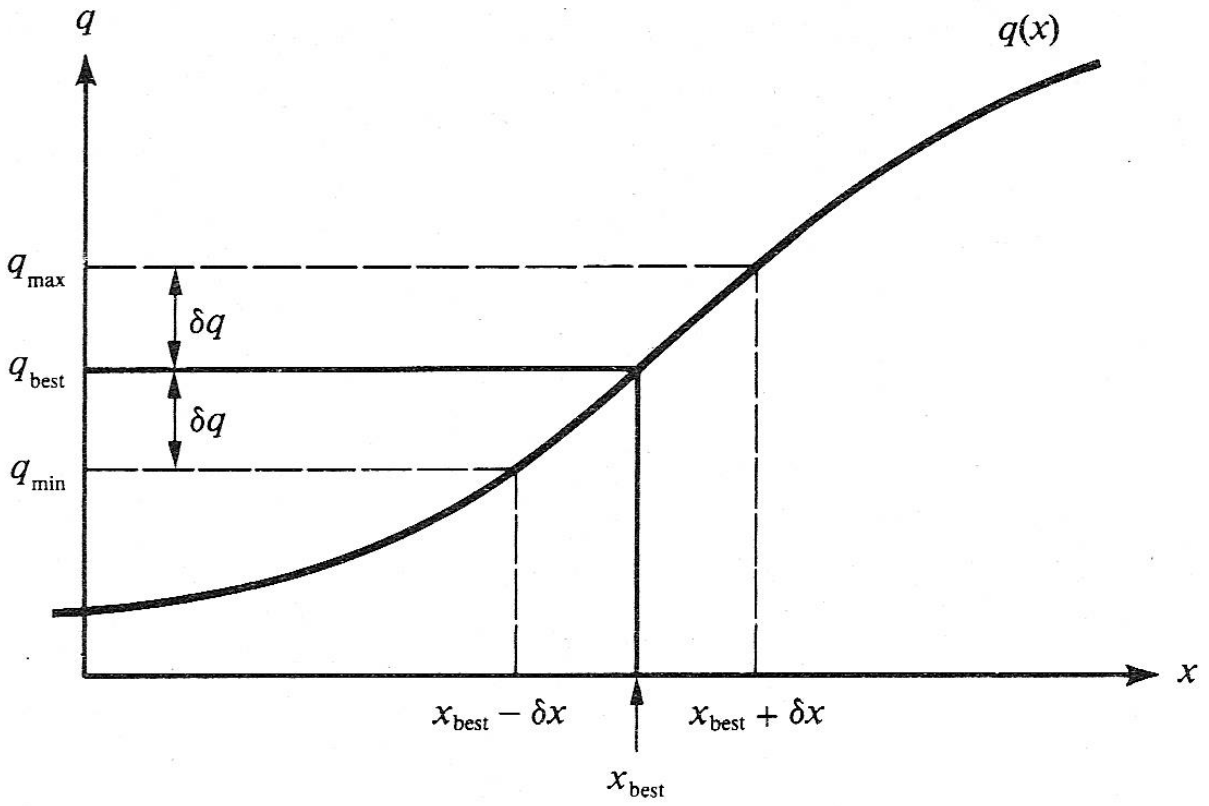
If this is the case, then $\delta q = \delta x + \delta y$ can be a correct choice otherwise, when the *measurements* and the *errors* of *x* and *y* are *independent*, there is always a "partial mutual deletion" of the uncertainties, the algebraic sum of the *x* and *y* errors is then an "*overestimation*" of the *q uncertainties* and it is much more reasonable to *add the errors for x and y in quadrature* :




$$\delta q = \sqrt{\delta x^2 + \delta y^2} < \delta x + \delta y$$

Similarly, when we have a measurement expressed as a ***product*** or a ***quotient*** of two primary quantities $q = x \cdot y$ the general rule is to add the *relative errors in quadrature* :

$$\frac{\delta q}{q} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta y}{y}\right)^2}$$




In general, when the variable of interest *q* is a <u>function of a measurable physical quantity *x*</u> : *q = q(x)*
*(for ex.* $q(x) = 1/senx$*)*

It is always possible to *measure* $x = x_b \pm \delta x$ and *calculate* $q_b = q(x_b)$ with the function relationship !
But how are we going to *calculate δq* ?

If *δx* is small and due only to *random errors*, $q_{min}$ and $q_{max}$ are almost equidistant from $q_b$ of a distance *δq*, regardless of the function type !

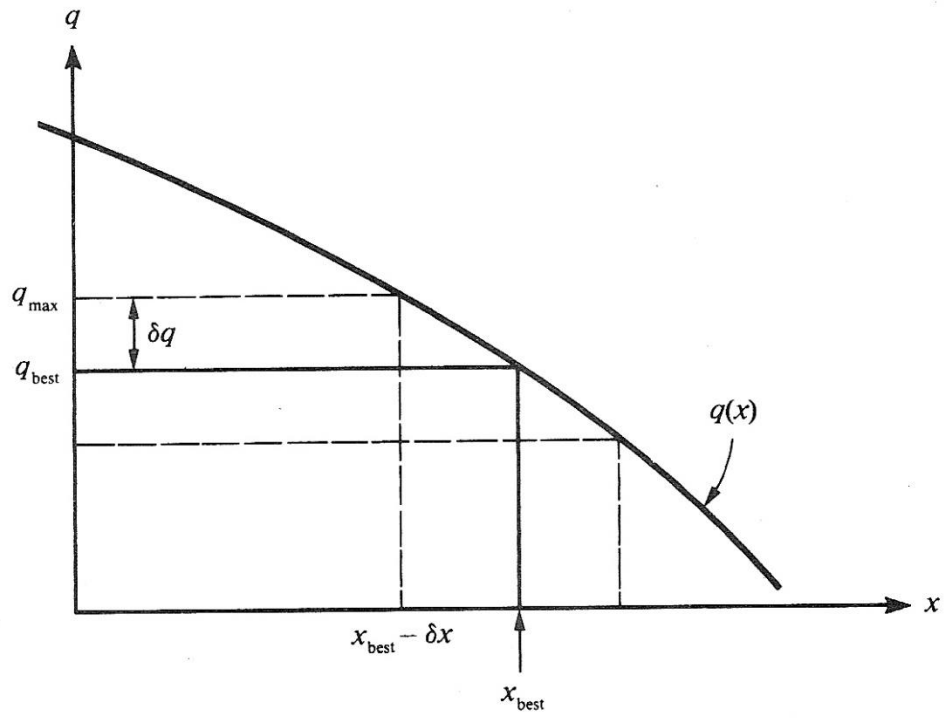
In this hypothesis we can write: $\delta q = q(x_b + \delta x) - q(x_b)$
which for *δx* → small, can also be written as:

$$\lim_{\delta x \to 0} \frac{\delta q}{\delta x} = \lim_{\delta x \to 0} \frac{q(x_b + \delta x) - q(x_b)}{\delta x} = \frac{dq}{dx} = \dot{q}(x_b)$$

which is the ***derivative*** of the function *q(x)* calculated in $x_b$

We reach then the important relationship:

$$\boxed{\delta q = \frac{dq}{dx} \cdot \delta x}$$




To include also the common cases when the function q(x) is *decreasing* in $x_b$ and has a *negative derivative* in the point $x_b$ :

$\frac{dq}{dx} < 0$ we should rather adjust the result and consider the ***absolute value of the derivative*** …

$$\delta q = \left|\frac{dq}{dx}\right| \cdot \delta x$$

In general, when we wish to know the *intensity of a physical quantity "q"* that can be expressed with a *function of two or more other physical variables* $q = q(x,y)$ , and we *measure these primary variables with their uncertainties*: $x = x_b \pm \delta x$ and $y = y_b \pm \delta y$ ; then we can always use the function relationship to calculate the *best representative* of $q$ : $q_b = q(x_b, y_b)$

while for $\delta q$ we might consider to apply the superimposition of the effects and use the algebraic sum:

$$\delta q = \left|\frac{\partial q}{\partial x}\right| \cdot \delta x + \left|\frac{\partial q}{\partial y}\right| \cdot \delta y$$

However, again, if the measurements of *x* and *y* and their errors are *independent*, it is quite reasonable to consider that there will be a "partial mutual deletion" of the uncertainties and, to express the general uncertainty $\delta q$ it is much more reliable to *add the uncertainties of x and y in quadrature:*

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x}\delta x\right)^2 + \left(\frac{\partial q}{\partial y}\delta y\right)^2}$$

Please, note that the *width parameters* we used for the *measurements x* and *y* are nothing more than the ***Standard Deviations*** previously calculated: $\delta x = \sigma_x$ and $\delta y = \sigma_y$

Everything said so far can be *generalized* for the case of a physical quantity $q = q(x_1, x_2, \ldots x_n)$ which is a *function of n measurable primary independent variables:*

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x_1}\sigma_{x_1}\right)^2 + \left(\frac{\partial q}{\partial x_2}\sigma_{x_2}\right)^2 + \ldots + \left(\frac{\partial q}{\partial x_n}\sigma_{x_n}\right)^2}$$

Which is the *fundamental statistical law* of the ***Uncertainty Propagation*** and represents also the *Combined Standard Uncertainty* for *uncorrelated input quantities* reported in the **GUM – JCGM 100: 2008**

We are now ready to answer our last question:
Can we actually *estimate somehow* the ***accuracy of the measurement*** itself with only a *"limited number N"* of *measurements* $x_1\, x_2 \ldots x_N$ ??
This goal can be approached by estimating "how well" the *mean value* $\bar{x}$ represents the *true value* $X$ or, trying to calculate the ***uncertainty of*** $\bar{x}$ when the mean value is the *best representative* of the true value $X$ .

To do so, we start dividing our ***N measurements*** $x_i$ in <u>*m groups*</u> of <u>*n measurement*</u> each :

$x_1' \, x_2' \ldots x_n' \rightarrow \bar{x}'$
$x_1'' \, x_2'' \ldots x_n'' \rightarrow \bar{x}''$
.....
$x_1^{(m)} \, x_2^{(m)} \ldots x_n^{(m)} \rightarrow \bar{x}^{(m)}$

Now we have $N = n \times m$ *measurements* and, for each group of m measurements, we can calculate the *mean value* $\bar{x}^j$ j = 1, 2 … m .

$$\bar{x}^j = \frac{1}{n}\sum_{i=1}^{n} x_i^j$$

Who is now the <u>*best representative*</u> of the *m mean values* $\bar{x}^j$ ??
It's the <u>*mean value of the m mean values* $\bar{x}^j$</u> :

$$\bar{\bar{x}} = \frac{1}{m}\sum_{j=1}^{m} \bar{x}^j$$

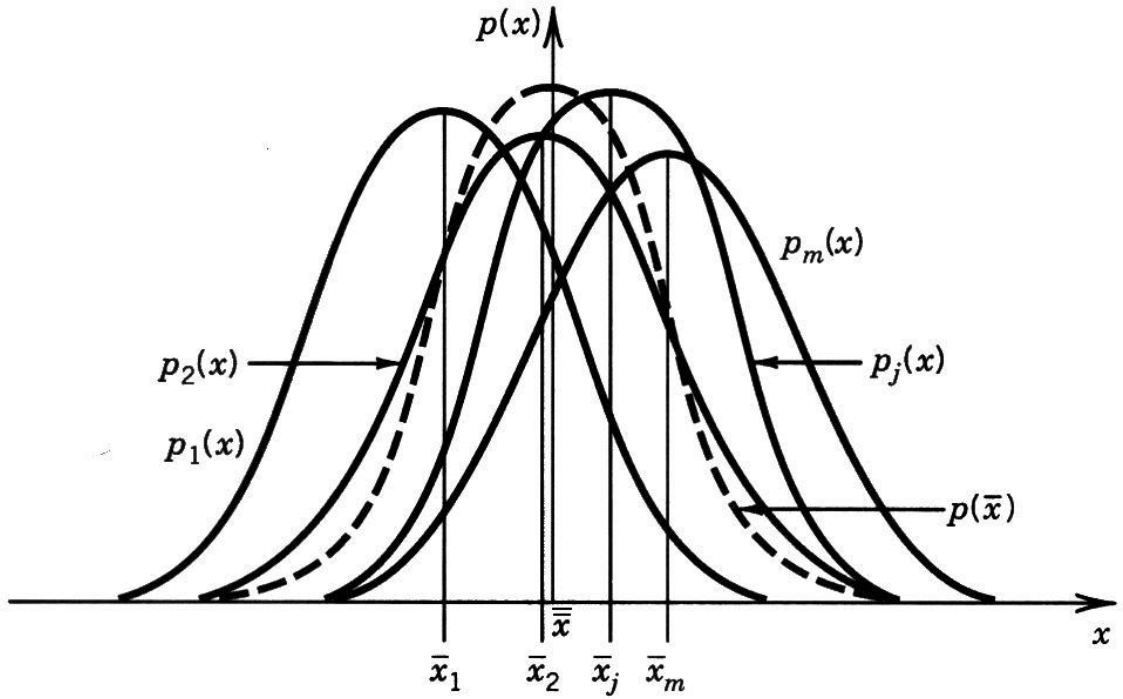
the ***"mean of the means"***

If $X$ is the "true value" of the ***n × m measurements*** $x_i^j$ , it is also the "true value" for the ***m means*** $\bar{x}^j$ because they come from the same measurements !

If the ***n × m measurements*** $x_i^j$ are affected only by small random errors, each *distribution curve* for ***the m groups*** of ***n measurements*** will be a ***normal (Gaussian) distribution*** curve $p_j(x)$ .
Therefore, the ***m mean values*** $\bar{x}^j$ will also distribute with a ***normal (Gaussain) curve*** $p(\bar{x})$ around the ***"mean of the means"***.

This happens because *each mean value* $\bar{x}^j$ is a *function of the n measurements* $x_i^j$ :

$$\bar{x}^j = \frac{1}{n}\sum_{i=1}^{n} x_i^j = f\left(x_i^j\right)$$




The *width parameter* $\delta\bar{x}$ for the distribution of the *m mean values* $\bar{x}^j$ will be :

$$\delta\bar{x} = \sigma_{\bar{x}} = \sqrt{\left(\frac{\partial\bar{x}}{\partial x_1}\delta x_1\right)^2 + \left(\frac{\partial\bar{x}}{\partial x_2}\delta x_2\right)^2 + \ldots + \left(\frac{\partial\bar{x}}{\partial x_n}\delta x_n\right)^2}$$

the ***Standard Deviation of the mean***

The width parameters $\delta x_i$ are here the ***Standard Deviations*** $\sigma_x^{(j)}$ calculated with the measurements of the $j_{th}$ *measurements group*. Since $x_i^j$ are all measurements of the *same quantity* done with the *same instrument* …

… for a *"**reasonable** number n" of measurements*, they will be almost coincident between them :

$$\sigma_x^{(1)} = \sigma_x^{(2)} = \ldots = \sigma_x^{(n)} = \sigma_x$$

From the definition of mean value: $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ it is easily obtained: $\frac{\partial \bar{x}}{\partial x_1} = \frac{\partial \bar{x}}{\partial x_2} = \ldots = \frac{\partial \bar{x}}{\partial x_n} = \frac{1}{n}$

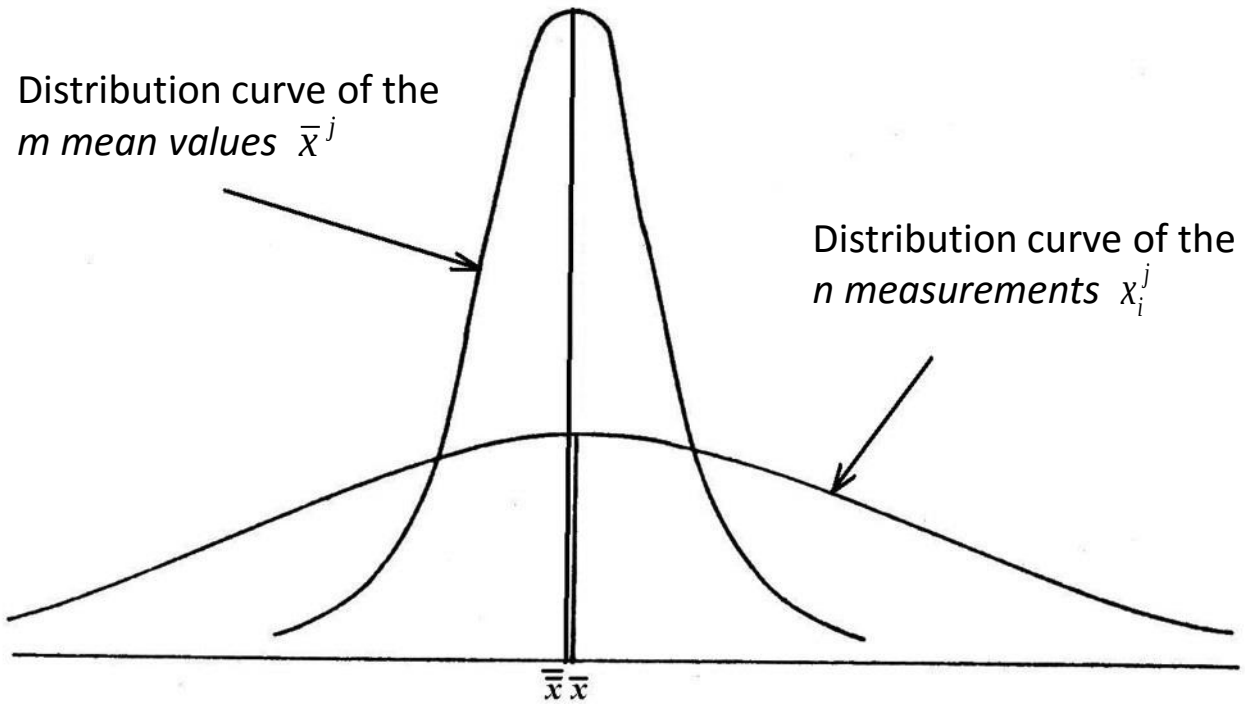
And substituting all these positions in the main equation of the width parameter it results:

$$\sigma_{\bar{x}} = \sqrt{\left(\frac{1}{n}\sigma_x\right)^2 + \left(\frac{1}{n}\sigma_x\right)^2 + \ldots + \left(\frac{1}{n}\sigma_x\right)^2} = \sqrt{n \cdot \frac{\sigma_x^2}{n^2}} = \frac{\sigma_x}{\sqrt{n}}$$

Which is the ***Standard Error** of the mean* and gives the ***uncertainty*** with which *the mean represents the "true value"* !

Please, carefully note:

- the *Standard Error* can be calculated only with the *n measurements of one single group* but, because of the way it was defined, it keeps the powerful meaning of ***UNCERTAINTY of the MEASUREMENT*** !!
- if we make *more measuements* (n → big) the *Standard Error decreases* while the *Standard Deviation stays about the same* ! … ***making more acquisitions makes a better measurement, NOT a better instrument*** !




For the same set on *n measurements*, the ***Standard Error*** is always “smaller” than the ***Standard Deviation*** !!

$$x_i \ldots x_{n+1} = \bar{x} \pm \sigma_x$$

***Standard Deviation*** expresses the ***instrument accuracy*** !

$$x = \bar{x} \pm \sigma_{\bar{x}}$$

***Standard Error*** expresses the ***measurement accuracy*** !